Finding reactions and displacements in indeterminate beams

In order to find displacements and reactions in indeterminate structures, it is necessary to use not only the shear and moment equations, but also the slope and displacement equations. This handout will cover how to find the reactions in an indeterminate beam by:

1. Integration and application of boundary conditions (tedious).
2. Use of beam tables and superposition.

Though more supports means more degrees of indeterminacy, it also means more boundary conditions, which are helpful in solving for constants. See figure 1 for the indeterminate beam that we are finding the reactions of.

![Indeterminate Beam Diagram](image)

**Figure 1: Indeterminate beam**

**Integration** - Integrate the load four times to get to displacements:

1. Integrate the load to get shear:

\[
V_1 = \int -w_1 dx_1 \\
V_1 = -wx_1 + C_1 \\
V_1(0) = R_A = C_1 \\
V_1 = -wx_1 + R_A \\
V_2 = \int 0 dx_2 \\
V_2 = C_2 = -R_B
\]

2. Integrate the shear to obtain moments:

\[
M_1 = \int V_1 dx_1 = \int (-wx_1 + R_A) dx_1 \\
M_1 = \frac{1}{2}wx_1^2 + R_A x_1 + C_1 \\
M_1(0) = -M_A = C_1 \\
M_1 = \frac{1}{2}wx_1^2 + R_A x_1 - M_A \\
M_2 = \int V_2 dx_2 = \int -R_B dx_2 \\
M_2 = -R_B x_2 + C_2 \\
M_2((1-a)L) = -R_B(1-a)L + C_2 = -M_B; \quad C_2 = -M_B + R_B(1-a)L \\
M_2 = -R_B x_2 - M_B + R_B(1-a)L
\]
3. Integrate the moments, divided by the flexural rigidity to obtain slopes:

\[
\theta_1 = \int \frac{M_1}{EI} \, dx_1 = \frac{1}{EI} \int \left( -\frac{1}{2} wx_1^2 + R_A x_1 - M_A \right) \, dx_1
\]  
(15)

\[
\theta_1 = \frac{1}{EI} \left( -\frac{1}{6} wx_1^3 + \frac{1}{2} R_A x_1^2 - M_A x_1 \right) + C_1
\]  
(16)

\[
\theta_1(0) = C_1 = 0
\]  
(17)

\[
\theta_1 = \frac{1}{EI} \left( -\frac{1}{6} wx_1^3 + \frac{1}{2} R_A x_1^2 - M_A x_1 \right)
\]  
(18)

\[
\theta_2 = \int \frac{M_2}{EI} \, dx_1 = \frac{1}{EI} \int \left[ -R_B x_2 - M_B + R_B (1-a) L \right] \, dx_2
\]  
(19)

\[
\theta_2 = \frac{1}{EI} \left( -\frac{1}{2} R_B x_2^2 + [-M_B + R_B (1-a) L] \, x_2 \right) + C_2
\]  
(20)

\[
\theta_2 ((1-a) L) = \frac{1}{EI} \left( -\frac{1}{2} R_B (1-a)^2 L^2 + [-M_B + R_B (1-a) L] (1-a) L \right) + C_2 = 0
\]  
(21)

\[
C_2 = \frac{1}{EI} \left[ -\frac{1}{2} R_B (1-a)^2 L^2 + M_B (1-a) L \right]
\]  
(22)

\[
\theta_2 = \frac{1}{EI} \left( -\frac{1}{2} R_B x_2^2 + [-M_B + R_B (1-a) L] \, x_2 - \frac{1}{2} R_B (1-a)^2 L^2 + M_B (1-a) L \right)
\]  
(23)

4. Integrate the slopes to obtain displacements:

\[
v_1 = \int \theta_1 \, dx_1
\]  
(25)

\[
v_1 = \frac{1}{EI} \left( -\frac{1}{24} wx_1^4 + \frac{1}{6} R_A x_1^3 - \frac{1}{2} M_A x_1^2 \right) + C_1
\]  
(26)

\[
v_1(0) = C_1 = 0
\]  
(27)

\[
v_1 = \frac{1}{EI} \left( -\frac{1}{24} wx_1^4 + \frac{1}{6} R_A x_1^3 - \frac{1}{2} M_A x_1^2 \right)
\]  
(28)

\[
v_2 = \int \theta_2 \, dx_2
\]  
(29)

\[
v_2 = \frac{1}{EI} \left( -\frac{1}{6} R_B x_2^3 + \frac{1}{2} [-M_B + R_B (1-a) L] \, x_2^2 \right) +
\]  
(30)

\[
v_2 = \frac{1}{EI} \left( \left[ -\frac{1}{2} R_B (1-a)^2 L^2 + M_B (1-a) L \right] \, x_2 \right) + C_2
\]  
(30)

\[
v_2 ((1-a) L) = \frac{1}{EI} \left( -\frac{1}{6} R_B (1-a)^3 L^3 + \frac{1}{2} [-M_B + R_B (1-a) L] (1-a)^2 L^2 \right) +
\]  
(31)

\[
v_2 = \frac{1}{EI} \left[ -\frac{1}{2} R_B (1-a)^2 L^2 + M_B (1-a) L \right] (1-a) L \right) + C_2 = 0
\]  
(32)

\[
C_2 = \frac{1}{EI} \left[ \frac{1}{6} R_B (1-a)^3 L^3 - \frac{1}{2} M_B (1-a)^2 L^2 \right]
\]  
(33)

\[
v_2 = \frac{1}{EI} \left( -\frac{1}{6} R_B x_2^3 + \frac{1}{2} [-M_B + R_B (1-a) L] \, x_2^2 \right) + \left[ -\frac{1}{2} R_B (1-a)^2 L^2 + M_B (1-a) L \right] \, x_2 +
\]  
(33)

\[
v_2 = \frac{1}{EI} \left( \frac{1}{6} R_B (1-a)^3 L^3 - \frac{1}{2} M_B (1-a)^2 L^2 \right)
\]  
(34)

5. At this point there are four unknowns - the moments and forces at the supports. Two equations can
be obtained from statics:

\[ \sum F_y = 0 = -waL + R_A + R_B \]  
\[ R_A + R_B = waL \]  
\[ \sum M_A = 0 = M_A - M_B + \frac{aL}{2} \quad (35) \]
\[ M_A - M_B + R_BL = \frac{1}{2}wa^2L^2 \]  
\[ (36) \]
Equations (36) and (38) will be two of the equations used to find the reactions.

6. Use the fact that the displacements and slopes must match where sections 1 and 2 meet in order to obtain two more equations:

\[ \theta_1(aL) = \theta_2(0) \]  
\[ \frac{1}{EI} \left[ -\frac{1}{6}wa^3L^3 + \frac{1}{2}R_Aa^2L^2 - MAaL \right] = \frac{1}{EI} \left[ -\frac{1}{2}R_B(1-a)^2L^2 + MB(1-a)L \right] \]  
\[ \frac{1}{2}R_Aa^2L + \frac{1}{2}R_B(1-a)^2L - MAa - MB(1-a) = \frac{1}{6}wa^3L^3 \]  
\[ v_1(aL) = v_2(0) \]  
\[ \frac{1}{6}R_Aa^3L - \frac{1}{6}(1-a)^3R_BL - \frac{1}{2}MAa^2 + \frac{1}{2}(1-a)^2MB = \frac{1}{24}wa^4L^2 \]  
\[ (37) \]

Equations (41) and (43) are the final two equations used to solve for the unknown reactions. The following linear system can be formed:

\[
\begin{bmatrix}
1 & 1 & 0 & 0 \\
\frac{1}{2}a^2L & \frac{1}{2}(1-a)^2L & -a & -(1-a) \\
\frac{1}{6}a^3L & \frac{1}{6}(1-a)^3L & -\frac{1}{2}a^2 & \frac{1}{2}(1-a)^2
\end{bmatrix}
\begin{bmatrix}
R_A \\
R_B \\
M_A \\
M_B
\end{bmatrix}
= \begin{bmatrix}
\frac{waL}{\frac{1}{2}wa^2L^2} \\
\frac{1}{2}wa^3L^2 \\
\frac{1}{24}wa^4L^2
\end{bmatrix}
\]

7. The reactions can be solved for in general by solving equation (44). The solution for a general value of ‘a’ is too cumbersome to write, therefore, the solution for the reactions when \( a = \frac{1}{2} \) is given below (and can be checked with beam tables):

\[
\begin{bmatrix}
R_A \\
R_B \\
M_A \\
M_B
\end{bmatrix} = \begin{bmatrix}
\frac{13}{12}wL \\
\frac{11}{12}wL \\
\frac{11}{12}wL^2 \\
\frac{11}{12}wL^2
\end{bmatrix}
\]

From equation (45), the displacements and angles can be determined.

**Superposition** - As illustrated by the above procedure, integration can be a very tedious, error prone process. If the indeterminate beam can be divided into simple beams, the simpler structures (which may have solutions in the beam tables) can be superimposed to find the reactions. See figure 2 for the division of beam into simpler structures. The superposition part is carried out assuming that \( a = \frac{1}{2} \), in order to simplify the process.

1. Looking in the beam tables for structure 1, the closest match is one with a distributed load along the entire length. This can be used to find the displacement at \( L/2 \) and then find the tip displacement from there. Since there is no shear or moment in the second half of the beam, the slope is constant. The slope is equal to the slope at the tip of the loaded section:

\[ \theta_{12} = -\frac{w}{6EI} \left( \frac{L}{2} \right)^3 = \theta_{1B} \]  
\[ (46) \]

The displacement in section two of structure 1 is determined as follows, using the fact that at the
beginning of section 2, the displacement is equal to that at the end of the loaded section:

\[ v_{12} = \int \theta_{12} \, dx_2 = -\frac{w}{6EI} \left( \frac{L}{2} \right)^3 x_2 + C_2 \]  

(47)

\[ v_{12}(0) = -\frac{w}{8EI} \left( \frac{L}{2} \right)^4 = C_2 \]  

(48)

\[ v_{1B} = -\frac{w}{6EI} \left( \frac{L}{2} \right)^3 \frac{L}{2} - \frac{w}{8EI} \left( \frac{L}{2} \right)^4 = -\frac{7wL^4}{384EI} \]  

(49)

2. The displacements and slopes in structure 2 are easy to determine from the beam tables:

\[ \theta_{2B} = \frac{R_B L^2}{2EI} \]  

(50)

\[ v_{2B} = \frac{R_B L^3}{3EI} \]  

(51)

3. The displacements and slopes in structure 3 are also easy to determine from the beam tables:

\[ \theta_{3B} = -\frac{M_B L}{EI} \]  

(52)

\[ v_{3B} = -\frac{M_B L^2}{2EI} \]  

(53)

4. Now, use the fact that the displacement and slopes at the right end are both zero in order to solve for \( R_B \) and \( M_B \):

\[ v_B = 0 = -\frac{7wL^4}{384EI} + \frac{R_B L^3}{3EI} - \frac{M_B L^2}{2EI} \]  

(54)

\[ \theta_B = 0 = -\frac{w}{6EI} \left( \frac{L}{2} \right)^3 + \frac{R_B L^3}{3EI} - \frac{M_B L^2}{2EI} \]  

(55)

5. If equations (54) and (55) are solved, the same value for \( R_B \) and \( M_B \) are obtained as with integration. The reactions at A can then be obtained by statics on the entire beam.