Handout 1 - 1st Order Stress Cube and Derivation of Shear Stress Equality

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Figure 1: 0th order view of stress (left) and 1st order view of stress (right)

If a body is cut by six planes at a point, an infinitesimal cube is produced. The left side of figure 1 shows and 0th order view of the internal stresses that are acting at the point. The cube axes are aligned with the principal axes. The lengths of the sides of the cubes are dx, dy, and dz, as indicated in figure 1.

The 0th order view representation, however, is not sufficient, as it does not consider changes in stress across the volume. A first order (remember Taylor expansions?) expansion of the stress state is shown on the right side of figure 1. Also, body forces per unit volume ($B$), for example gravity, are shown acting in the principal directions at the center of the cube.

Goal: Demonstrate equivalence of shear stresses: $\tau_{ij} = \tau_{ji}$, which means that $\tau_{xy} = \tau_{yx}$, $\tau_{xz} = \tau_{zx}$ and $\tau_{yz} = \tau_{zy}$.

1. Sum moments about the z-axis:

$$\sum M_z = 0 = -(\sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x} dx) \frac{dy dz}{2} + \sigma_{xy} dy dz \frac{dy}{2} + (\sigma_{yy} + \frac{\partial \sigma_{yy}}{\partial y} dy) dz dx \frac{dx}{2} - \sigma_{yx} dz dy \frac{dx}{2} +$$

$$\begin{array}{c}
(t_{xy} + \frac{\partial t_{xy}}{\partial x} dx) dy dz \frac{dx}{2} - (t_{yx} + \frac{\partial t_{yx}}{\partial y} dy) dz dx \frac{dy}{2} - (t_{zz} + \frac{\partial t_{zz}}{\partial z} dz) dx dy \frac{dy}{2} + \\
\tau_{xx} dx dy \frac{dy}{2} + (\tau_{xy} + \frac{\partial \tau_{xy}}{\partial z} dz) dz dx \frac{dx}{2} - \tau_{yx} dz dy \frac{dx}{2} - Bz \frac{dy dz dx}{2} + B_y dz dy dx \frac{dx}{2} \\
\end{array}$$

$\cdot$ Forces acting directly on or along the z-axis cause no moments.
2. **Ignore all quadruple products**: All terms in equation (1) have at least a triple product with some combination of \( dx \), \( dy \) and \( dz \). The derivative terms have a quadruple product. Since these are infinitesimal quantities, the quadruple product is insignificant compared to the triple product. Therefore, all of the terms that involve a quadruple product are considered to be zero. Equation (1) is repeated below with those terms canceled:

\[
\sum M_z = 0 = -(\sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x} dx)dydz \frac{dy}{2} + \sigma_{xx} dydz \frac{dy}{2} + (\sigma_{yy} + \frac{\partial \sigma_{yy}}{\partial y} dy)dzdx \frac{dx}{2} - \sigma_{yy} dzdx \frac{dx}{2} + \\
(\tau_{xy} + \frac{\partial \tau_{xy}}{\partial x} dx)dydz = (\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y} dy)dzdy - (\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z} dz)dzdy \frac{dy}{2} + \\
\tau_{xx} dzdy \frac{dy}{2} + (\tau_{zy} + \frac{\partial \tau_{zy}}{\partial z} dz)dzdy \frac{dx}{2} - \tau_{zy} dzdy \frac{dx}{2} - B_x dzdydz \frac{dy}{2} + B_y dzdydz \frac{dx}{2}
\]

(2)

- The results in the following remainder terms:

\[
\sum M_z = 0 = -\sigma_{xx} \frac{dy^2 dz}{2} + \sigma_{xx} \frac{dy^2 dz}{2} + \sigma_{yy} \frac{dz^2 dx}{2} - \sigma_{yy} \frac{dz^2 dx}{2} + \\
\tau_{xy} dzdydz - \tau_{yx} dzdydz - \tau_{xx} \frac{dx^2 dy}{2} + \tau_{xx} \frac{dx^2 dy}{2} + \\
\tau_{zy} \frac{dx^2 dy}{2} - \tau_{zy} \frac{dx^2 dy}{2}
\]

(3)

3. **Simplify**: From equation (3) it is possible to make several simplifications due to canceling terms. Equation (4) repeats equation (3) with the appropriate cancellations:

\[
\sum M_z = 0 = -\sigma_{xx} \frac{dy^2 dz}{2} + \sigma_{xx} \frac{dy^2 dz}{2} + \sigma_{yy} \frac{dz^2 dx}{2} - \sigma_{yy} \frac{dz^2 dx}{2} + \\
\tau_{xy} dzdydz - \tau_{yx} dzdydz - \tau_{xx} \frac{dx^2 dy}{2} + \tau_{xx} \frac{dx^2 dy}{2} + \\
\tau_{zy} \frac{dx^2 dy}{2} - \tau_{zy} \frac{dx^2 dy}{2}
\]

(4)

- Only the following terms remain:

\[
\tau_{xy} dzdydz - \tau_{yx} dzdydz = 0
\]

(5)

- From equation (5) it can easily be seen that:

\[
\tau_{xy} = \tau_{yx}
\]

(6)

- Equation (6) is the goal of this handout. The same thing can be demonstrated for the other shear stresses, one of which is the subject of a problem in homework 1.